Sampling with Time-Changed Markov Processes

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4 December 2025,
Lancaster CSML

Outline

Goal and Motivation

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- We suggest a general framework for algorithms to speed up or slow down in various areas.
- Follows up work from (Vasdekis and Roberts 2023).

Challenges with Traditional MCMC

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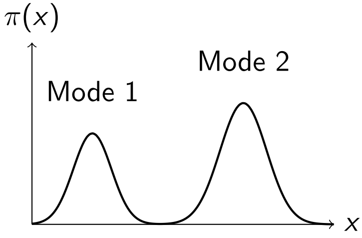

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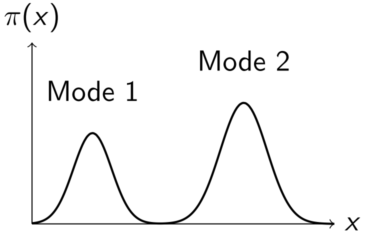

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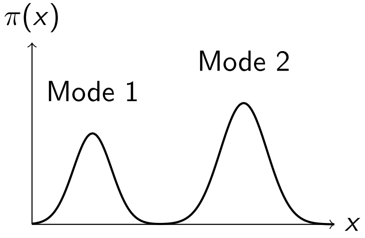

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Need for More Efficient MCMC Methods

Traditional MCMC methods often struggle with these challenging distributions, leading to slow convergence and poor mixing.

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- The transformation is regulated by a **speed function** $s : \mathbb{R}^d \to (0, +\infty)$.
- Intuitively:
  - When $s(x)$ is large, time accelerates
  - When $s(x)$ is small, time decelerates

Mathematical Formulation

Time-Changed Process Definition

$$X_t = Y_{r(t)}$$

where

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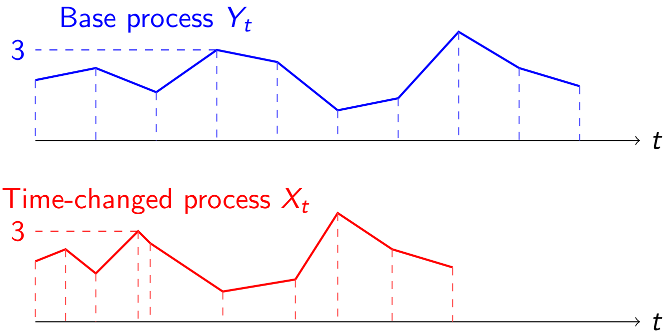
"X follows the path of Y, but s times faster."

Visualizing Time-Changed Processes



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- Lower bounded: $s(x) \geq s_0 > 0$.

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Informal Result: Invariant distributions

Let $\tilde{\pi}(dx) = \frac{1}{\pi(s)} s(x)\pi(dx)$.

- Y targets $\tilde{\pi} \iff X$ targets π.

where $\pi(s) = \int_{\mathbb{R}^d} s(x)\pi(dx)$

Example: Multimodal Distribution

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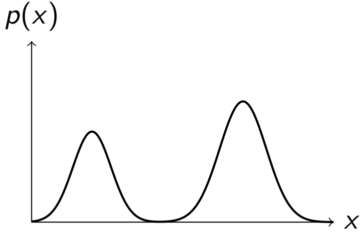

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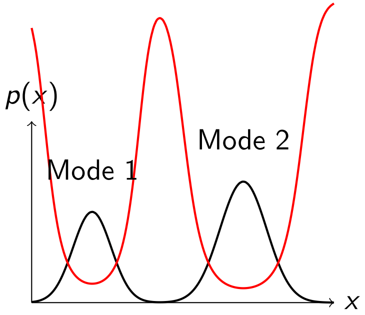
Solution with Time-Change:

- Set high $s(x)$ in low-density regions
- Process spends less real time there
- But visits these regions more frequently
- Improves mode-hopping behavior



Key Assumptions

Assumption 1 (Speed Function)

The speed function $s : \mathbb{R}^d \to \mathbb{R}_+$ is continuous, satisfies $\int s(y)\pi(dy) < \infty$, and there exists $s_0 > 0$ such that $s(x) \geq s_0$ for all $x \in \mathbb{R}^d$.

Assumption 2 (LLN for Base Process)

For any $f \in L^1(\tilde{\pi})$ and any initial condition $x \in E$:

$$\frac{1}{T}\int_0^T f(Y_u)du \xrightarrow{T\to\infty} \int_E f(y)\tilde{\pi}(dy) \quad \text{a.s.} \tag{1}$$

where $\tilde{\pi}(dx) = \frac{1}{\pi(s)}s(x)\pi(dx)$.

Law of Large Numbers for X_t

Theorem (Invariance and LLN for Time-Changed Process)

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Under Assumptions 1 and 2, the process X has π as unique stationary distribution. Furthermore, for any $f \in L^1(\pi)$ and all initial conditions $x \in \mathbb{R}^d$:

$$\frac{1}{T}\int_0^T f(X_t)dt \xrightarrow{T\to\infty} \int_{\mathbb{R}^d} f(x)\pi(dx) \quad \text{a.s.} \tag{2}$$

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Convergence Properties

Geometric Ergodicity

Under suitable conditions on the base process Y and speed function s, the time-changed process X_t is geometrically ergodic, even if the base process Y is not:

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Uniform Ergodicity

Under suitable conditions on Y and assuming that $s(x)$ grows sufficiently fast as $\|x\| \to \infty$, the time-changed process X can be uniformly ergodic even when the base process Y is not:

$$\|\mathbb{P}(X_t \in \cdot) - \pi(\cdot)\|_{TV} \leq M\exp\{-\lambda t\}, t \geq 0$$

Convergence Properties

Central Limit Theorem

Under appropriate conditions, the time-changed process satisfies a central limit theorem, with asymptotic variance expressed in terms of the base process.

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- (Roberts and Stramer 2002)

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- The process $dX_t = s(X_t)\nabla \log\left(s(Y_t)\pi(Y_t)\right) dt + \sqrt{2}s^{1/2}(X_t)dW_t$ targets π.

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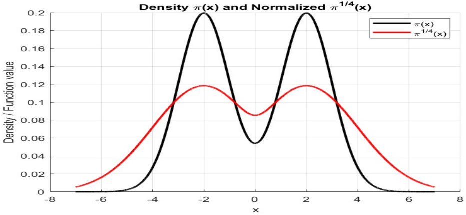

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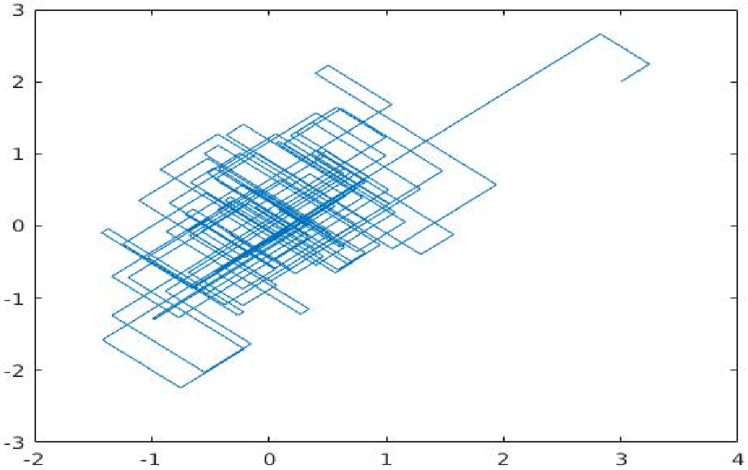
Running Example: Zig-Zag Process (ZZP)



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Running Example: Zig-Zag Process (ZZP)

Algorithm (in 2-d) targeting π

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Time-Changed Zig-Zag Process

Algorithm (in 2-d) targeting π

1. Start from point $(x, v) = (x_1, x_2; v_1, v_2) \in \mathbb{R}^2 \times \{-1, 1\}^2$.
2. The process (X_t, V_t) moves according to the deterministic dynamics:

$$\frac{d}{dt} X_t = s(X_t) \cdot v, \quad t \geq 0, \quad X_0 = x, \quad \text{and} \quad V_t = v, \quad t \geq 0.$$

3. For all $i = 1, 2$, consider a non-homogeneous Poisson process with intensity

$$m_i(t) = \lambda_i(X_t, v), \quad t \geq 0, \quad \text{where} \quad \lambda_i(x, v) = s(x) \left[-v_i \partial_i \log\left(s(x) \pi(x) \right) \right]^+ .$$

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$$c_2 = \sqrt{\frac{1 + y_2^2}{d} - c_1^2}, \quad c_1 = \frac{(y \cdot v)}{d} v_1, \quad y_2 = x_2 - v_1 v_i x_1.$$

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Diffeomorphic Transformations

Tempering/Annealing

- Modifies target distribution → Targets $\tilde{\pi}(x) = \pi^{\alpha}(x)$, $\alpha \in (0, 1)$

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- Both time and diffeomorphic transformations can achieve uniform ergodicity under suitable conditions

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- *For example:* Time-changed Zig-Zag process with speed

$$s(x) = \left(1 + \|x\|^2\right)^{1+k}, k = 0, 1, 2, \ldots$$

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Connection with **Importance Markov Chain.**

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- Easier to implement in some cases.
- $\tilde{Q}$ can be a discretisation of your favourite process.

Simulations: Heavy-Tailed Targets (Scenario 1)

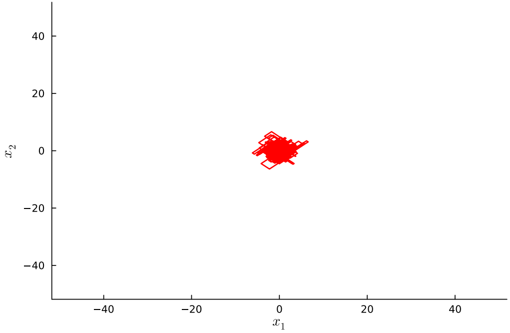

Figure: $s(x) = 1$

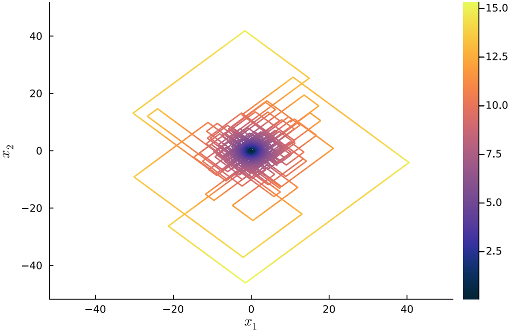

Figure: $s(x) = (1 + |x|^2)^2$

Figure: Student(5) target.

Simulations: Heavy-Tailed Targets (Scenario 2)

Target: $\pi(x) \sim \exp\left\{-\|x\|^{1/2}\right\}, \quad d = 20$

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Algorithmic Efficiency

Algorithms	**ESS/Lik.Eval.**	**ESS/min**
Zig-Zag	$0.3 \cdot 10^{-3}$	124.9
Time-changed ZZ ($a = 0$)	$3.9 \cdot 10^{-3}$	2847.7
Time-changed ZZ ($a = 1$)	$\mathbf{6.3 \cdot 10^{-3}}$	**4471.3**
Space Transformed RWM	$2.8 \cdot 10^{-3}$	1966.8

ESS for Zig-Zag, Time-transformed Zig-Zag, and Space Transformed Random Walk Metropolis

Simulations: Heavy-Tailed Targets (Scenario 3)

Speed function: $s(x) = \pi(x)^{-a}, \quad a \in (0, 1/3).$

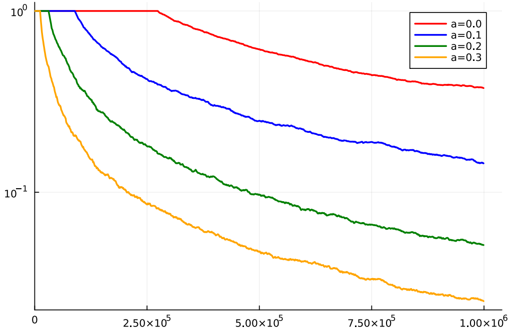

Figure: Median of the relative square error (y-axis) vs number of jumps of the base process (x-axis).

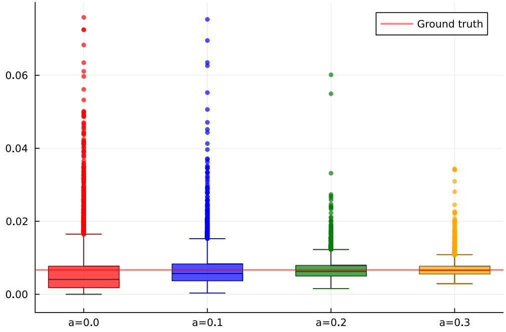

Figure: Estimates of $\mathbb{P}(\|x\| > 150)$ for different values of a.

Figure: Target: Student(1) distribution in $\mathbb{R}^2$.

Simulations: Multi-Modal Targets

Mixture of Normals

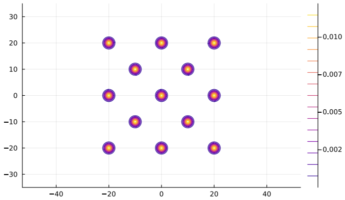

Figure: Level curves of π.

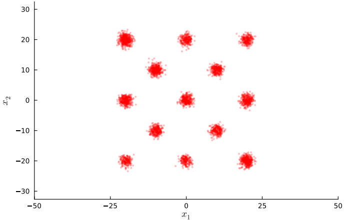

Figure: Time-changed: $s(x) = \pi(x)^{-0.9}$.

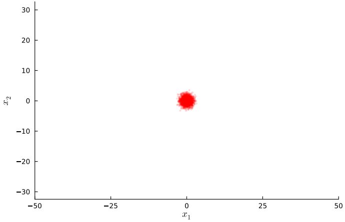

Figure: Without time-change: $s(x) = 1$.

Summary and Future Work

Future Research Directions

- Optimal choice of speed function for specific targets.
- Adaptive methods to learn optimal speed functions.
- Study the high-dimensional behaviour/ scaling limits.

References

- Bertazzi, A., & Vasdekis, G. (2025). Sampling with time-changed Markov processes. *arXiv preprint arXiv:2501.15155*. Under Revision.
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Thank you for your attention!

Paper: Sampling with time-changed Markov processes
Available at: `https://arxiv.org/abs/2501.15155`